PHYS 102 – General Physics II Midterm Exam 2

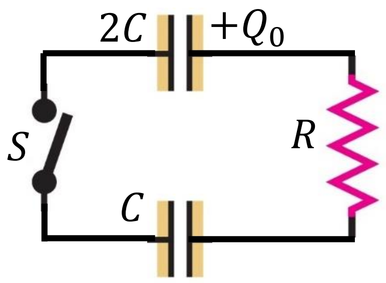
Duration: 120 minutes Saturday, 26 November 2022; 13:30

1. Consider the circuit given. Initially, switch S is open, the capacitor $2C$ has charge Q_0, and the capacitor C is uncharged. The switch S is closed at time $t = 0$.

(a) (5 Pts) What is the charge on each capacitor after a very long time.

(b) (15 Pts.) What is the current in the loop as a function of time for $t > 0$?

(c) (5 Pts.) What is the total energy dissipated in the resistor?



Solution:

(a) After the switch is closed some charge will flow from the charged capacitor to the uncharged capacitor until the potential differences on the two capacitors are equal.

$$V_{2C} = \frac{Q_{2C}}{2C}\,, \qquad V_C = \frac{Q_C}{C}\,, \qquad V_{2C} = V_C \quad \to \quad Q_{2C} = 2Q_C\,.$$

Total charge on the two capacitors will be equal to the initial charge.

$$Q_{2C} + Q_C = Q_0 \quad \to \quad 3Q_C = Q_0 \quad \to \quad Q_C = \frac{Q_0}{3}\,, \qquad Q_{2C} = \frac{2Q_0}{3}.$$

(b) Writing the loop equation, and noting that at any time $q_{2C}(t) + q_2(t) = Q_0$, we have

$$V_{2C} - Ri - V_C = 0\,, \qquad \frac{q_{2C}}{2C} - Ri - \frac{q_C}{C} = 0 \quad \to \quad \frac{Q_0}{2C} - Ri - \frac{3}{2C}q_C = 0\,.$$

Since $i = dq_C/dt$, above result can be rearranged as

$$\frac{dq_C}{q_C - Q_0/3} = -\frac{3}{2RC}dt \quad \to \quad \int_0^q \frac{dq_C}{q_C - Q_0/3} = -\frac{3}{2RC}\int_0^t dt' \quad \to \quad q(t) = \frac{Q_0}{3}\left(1 - e^{-3t/2RC}\right).$$

$$i(t) = \frac{dq}{dt} \quad \to \quad i(t) = \frac{Q_0}{2RC}e^{-3t/2RC}\,.$$

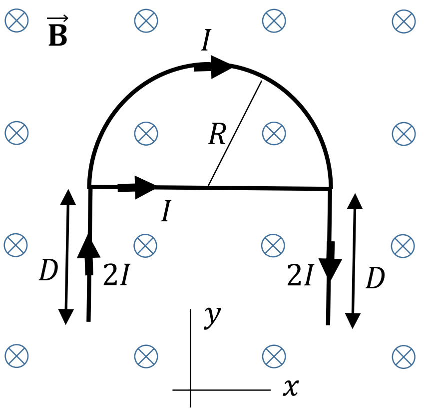
(c) Total energy dissipated in the resistor can be calculated in two different ways. One can integrate the power

$$U = \int_0^\infty Ri^2 dt = \frac{Q_0^2}{4RC^2}\int_0^\infty e^{-3t/RC}dt \quad \to \quad U = \frac{Q_0^2}{12C},$$

or one can calculate the difference in the energy stored in the capacitors

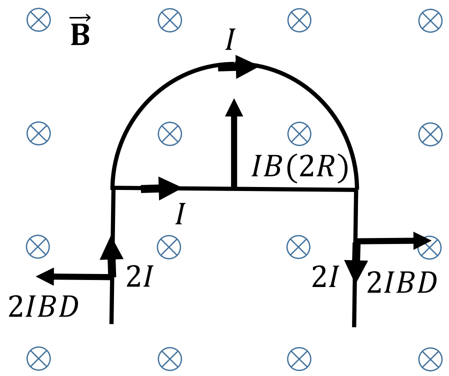
$$\Delta U = U_f - U_i = \frac{(Q_0/3)^2}{C} + \frac{(2Q_0/3)^2}{2C} - \frac{Q_0^2}{4C} \quad \to \quad \Delta U = -\frac{Q_0^2}{12C}$$

2. A rigid conductor consists of a semi-circular part of radius R carrying a current I, a straight part along a diameter carrying a current I, and two straight segments of length D carrying a current $2I$, in directions indicated in the figure. The conductor is placed inside a uniform magnetic field of magnitude B whose direction is perpendicular to the plane of the conductor, into the plane of the figure. Find the total force on the conductor. Use the coordinate system given in the figure.

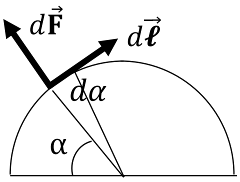


Solution:

Force on the straight segments can be found using $\vec{\mathbf{F}} = I\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$.



The force on the semicircular segment can be found by integration.



$$dF_x = -IRB\cos\alpha \; d\alpha \,,$$

$$dF_y = IRB\sin\alpha \; d\alpha \,.$$

$$F_x = -IRB\int_0^{\pi} \cos\alpha \; d\alpha = 0 \,,$$

$$F_y = IRB\int_0^{\pi} \sin\alpha \; d\alpha = 2IRB \,.$$

The net force on the conductor is

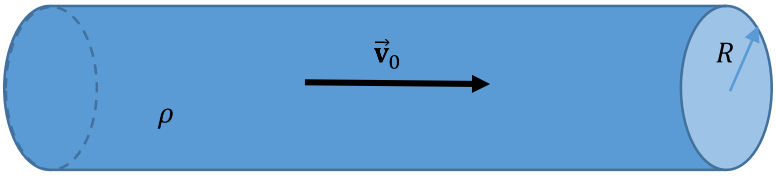
$$\vec{\mathbf{F}} = 4IBR\,\hat{\mathbf{j}} \,.$$

3. Assume that a beam of protons (elementary particles of mass m and charge e) forms a very long cylinder of radius R and uniform charge density ρ. Assume that all the protons are moving along the cylinder's axis with the same velocity $\vec{\mathbf{v}}_0$. You can regard the beam as infinitely long.

(a) (5 Pts.) What is the electrical current carried by the beam?

(b) (5 Pts.) What is the magnitude of the magnetic field on the surface of the beam?

(c) (5 Pts.) What is the magnitude of the electric field on the surface of the beam?



(d) (5 Pts.) Find the electrical force (direction and magnitude) and the magnetic force (direction and magnitude) acting on a proton of the beam, which is on the surface of the cylinder.

(e) (5 Pts.) Calculate the speed of the beam v_0 so that the two forces cancel each other and the radius of the beam does not expand with time.

Solution: (a)

$$I = \vec{\mathbf{J}} \cdot \vec{\mathbf{A}} = (\rho v_0)(\pi R^2) = \pi \rho R^2 v_0 \,.$$

(b) Magnetic field on the surface of the beam can be found using Ampère's law.

$$B(R) = \frac{\mu_0 I}{2\pi R} \quad \rightarrow \quad B(R) = \frac{1}{2}\mu_0 \rho R v_0 \,.$$

(c) Electric field on the surface of the beam can be found using Gauss's law.

$$E(R) = \frac{\lambda}{2\pi\epsilon_0 R} \quad \rightarrow \quad E(R) = \frac{\rho A}{2\pi\epsilon_0 R} = \frac{\rho \pi R^2}{2\pi\epsilon_0 R} \quad \rightarrow \quad E(R) = \frac{\rho R}{2\epsilon_0} \,.$$

(d) Electrical force on a proton at the surface of the beam is radially outward with magnitude

$$F_E = eE(R) \quad \rightarrow \quad F_E = \frac{e\rho R}{2\epsilon_0} \,.$$

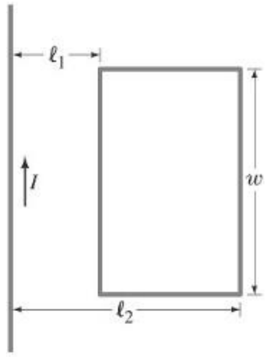
Magnetic force on a proton at the surface of the beam is radially inward with magnitude

$$F_B = ev_0 B(R) \quad \rightarrow \quad F_B = \frac{1}{2}\mu_0 \rho e R v_0^2 \,.$$

(e)

$$F_E = F_B \quad \rightarrow \quad \frac{e\rho R}{2\epsilon_0} = \frac{1}{2}\mu_0 \rho e R v_0^2 \quad \rightarrow \quad \mu_0 v_0^2 = \frac{1}{\epsilon_0} \quad \rightarrow \quad v_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c \,.$$

4. A very long straight wire carrying a current $I(t)$ and a small rectangular wire loop of resistance R lie in the same plane as shown in the figure.

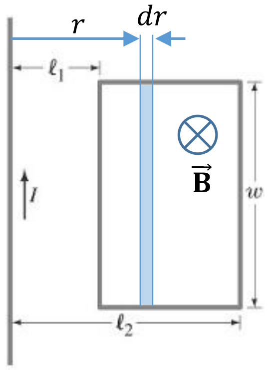


(a) (8 Pts.) Find the magnetic flux through the loop.

(b) (9 Pts.) If the current through the long straight wire changes in time according to $\frac{dI}{dt} = \alpha$ $(\alpha > 0)$, what current will be induced in the rectangular loop? Is the induced current clockwise of counter clockwise?

(c) (8 Pts.) What will be net force on the rectangular loop? Is the force attractive or repulsive? Explain why.

Solution:



(a) The magnetic field created by the long straight wire is perpendicular to the plane of the loop and is directed inward. Magnitude of the magnetic field depends on the perpendicular distance r measured from the line as

$$B(r) = \frac{\mu_0 I}{2\pi r}.$$

Magnetic flux through the shaded strip of infinitesimal width dr at a distance r is

$$d\Phi = B(r)w dr \quad \rightarrow \quad d\Phi = \frac{\mu_0 I w}{2\pi}\frac{dr}{r} \quad \rightarrow \quad \Phi = \frac{\mu_0 I w}{2\pi}\int_{\ell_1}^{\ell_2}\frac{dr}{r} \quad \rightarrow \quad \Phi = \frac{\mu_0 I w}{2\pi}\ln\left(\frac{\ell_2}{\ell_1}\right).$$

(b) Since the magnetic field strength is increasing, in accordance with Lenz's law, the induced current should be counter clockwise to oppose the increase. Its magnitude is found as

$$I_{\text{ind}} = \frac{|\mathcal{E}_{\text{ind}}|}{R}, \qquad |\mathcal{E}_{\text{ind}}| = \frac{d\Phi}{dt} = \frac{\mu_0 w}{2\pi}\ln\left(\frac{\ell_2}{\ell_1}\right)\frac{dI}{dt} \quad \rightarrow \quad I_{\text{ind}} = \frac{\mu_0 w \alpha}{2\pi R}\ln\left(\frac{\ell_2}{\ell_1}\right).$$

(c) Net force on the top and bottom segments of the loop is zero. For the side segments the magnitude of the net force is

$$F = \frac{\mu_0 I I_{\text{ind}} w}{2\pi\ell_1} - \frac{\mu_0 I I_{\text{ind}} w}{2\pi\ell_2} = \frac{\mu_0 I I_{\text{ind}} w}{2\pi}\left(\frac{1}{\ell_1} - \frac{1}{\ell_2}\right),$$

$$F = \frac{{\mu_0}^2 w^2 \alpha I}{4\pi^2 R}\left(\frac{\ell_2 - \ell_1}{\ell_1\ell_2}\right)\ln\left(\frac{\ell_2}{\ell_1}\right),$$

repelling the loop.